Stone spectra of finite von Neumann algebras of type $\mathrm{I_n}$

Hans F. de Groote

May 2, 2006

Abstract

In this paper, we clarify the structure of the Stone spectrum of an arbitrary finite von Neumann algebra $\mathcal{R}$ of type $\mathrm{I_n}$. The main tool for this investigation is a generalized notion of rank for projections in von Neumann algebras of this type.

1 Introduction

The Stone spectrum $\mathcal{Q}(\mathbb{L})$ of a lattice $\mathbb{L}$ is the set of all *maximal dual ideals* in $\mathbb{L}$, endowed with the following topology: For $a \in \mathbb{L}$, let

$$\mathcal{Q}_a(\mathbb{L}) := \{\mathfrak{B} \in \mathbb{L} \mid a \in \mathfrak{B}\}.$$

Then

$$\mathcal{Q}_0(\mathbb{L}) = \emptyset,\ \mathcal{Q}_1(\mathbb{L}) = \mathbb{L} \textit{ and } \mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_b(\mathbb{L}) = \mathcal{Q}_{a\wedge b}(\mathbb{L}) \textit{ for all } a, b \in \mathbb{L}.$$

Here 0 denotes the minimal and 1 the maximal element of $\mathbb{L}$. We assume here that such elements exist in $\mathbb{L}$.[1] Hence the sets $\mathcal{Q}_a(\mathbb{L}),\ a \in \mathbb{L}$, form a basis of a topology for $\mathcal{Q}(\mathbb{L})$. We call $\mathcal{Q}(\mathbb{L})$, together with this topology, the Stone spectrum of the lattice $\mathbb{L}$. This is an obvious generalization of Stone's construction ([16]). Stone's motivation was to represent an abstract Boolean algebra by a Boolean algebra of sets. However, there are two other scenarios, quite different from Stone's, that lead to the same construction.

If $\mathcal{S}$ is a presheaf on a complete lattice $\mathbb{L}$, we can define *germs* of $\mathcal{S}$ in every *filter base* in $\mathbb{L}$, i.e. in every subset $\mathcal{F}$ that satisfies

(i) $0 \notin \mathcal{F}$,

(ii) if $a, b \in \mathcal{F}$, there is $c \in \mathcal{F}$ such that $c \leq a, b$,

[1] This is not an essential restriction. If $\mathbb{L}$ is complete, they exist anyway: $0 = \bigwedge_{a\in\mathbb{L}} a$ and $1 = \bigvee_{a\in\mathbb{L}} a$.

as elements of the inductive limit

$$\varinjlim_{a \in \mathcal{F}} \mathcal{S}(a).$$

Of course, if $a \in \mathbb{L} \setminus \{0\}$, $\{a\}$ is a filter base, but in general it makes no sense to regard $\mathcal{S}(a)$ as a germ. Moreover, the set of *all* filter bases is a vast object. For the definition of an etale space corresponding to the presheaf $\mathcal{S}$, it is therefore meaningful to consider germs in maximal filter bases. This is why we speak of *quasipoints* instead of maximal filter bases. Now it is easy to see that maximal filter bases are nothing else but maximal dual ideals in $\mathbb{L}$. The *sheafification* of $\mathcal{S}$ is therefore defined over the Stone spectrum of the lattice $\mathbb{L}$ ([3]).

The Stone spectrum $\mathcal{Q}(\mathcal{R})$ of a von Neumann algebra $\mathcal{R}$ is by definition the Stone spectrum of its projection lattice $\mathcal{P}(\mathcal{R})$. If $\mathcal{R}$ is abelian, $\mathcal{Q}(\mathcal{R})$ is homeomorphic to the Gelfand spectrum of $\mathcal{R}$ ([3]). The Stone spectrum of $\mathcal{R}$ is therefore a generalization of the Gelfand spectrum to the non-abelian case. Moreover, also the Gelfand transformation has a natural generalization to the non-abelian case. If A is a selfadjoint element of a von Neumann algebra $\mathcal{R}$, and if $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral family of A, we define

$$f_A(\mathfrak{B}) := \inf\{\lambda \mid E_\lambda \in \mathfrak{B}\}$$

for all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$. The function $f_A : \mathcal{Q}(\mathcal{R}) \to \mathbb{R}$, called the *observable function corresponding to* A, is continuous and coincides with the Gelfand transform of A if $\mathcal{R}$ is abelian ([4]). This means that the Stone spectrum of $\mathcal{R}$ is a natural generalization of the Gelfand spectrum.
The Gelfand spectrum of an abelian von Neumann algebra is, in general, a rather wild object. So it is to be expected that the Stone spectrum of a non-abelian von Neumann algebra can have a very intricate structure.

In the case of $\mathcal{R} = \mathcal{L}(\mathbb{C}^n)$ however, the situation is quite simple: let $\mathfrak{B}$ be a quasipoint of $\mathcal{R}$ and let $P_0 \in \mathfrak{B}$ be a projection whose rank is minimal in the set $\{rk(P) \mid P \in \mathfrak{B}\}$. Pick a subprojection Q of P_0 that has rank one. Then $rk(P \wedge P_0) \le rk(P_0)$ and $P \wedge P_0 \in \mathfrak{B}$ for all $P \in \mathfrak{B}$, hence $P_0 \le P$ for all $P \in \mathfrak{B}$. This implies $Q \le P$ for all $P \in \mathfrak{B}$ and, therefore, $Q \in \mathfrak{B}$ by the maximality of $\mathfrak{B}$. Hence

$$\mathfrak{B} = \{P \in \mathcal{P}(\mathcal{R}) \mid Q \le P\}$$

for a unique $Q \in \mathcal{P}(\mathcal{R})$ of rank one. Since in a factor the abelian projections are minimal, the foregoing result can be expressed as: each quasipoint

contains an abelian projection.

If $\mathcal{R}$ is a von Neumann algebra of type $\mathrm{I_n}$, $\mathrm{n} \in \mathbb{N}$, with center $\mathcal{C}$, the above argument is not applicable since the rank of a projection is infinite in general. Nevertheless the result that each quasipoint of $\mathcal{R}$ contains an abelian projection is still true[2]. Even the simple idea of the foregoing proof is transferable - provided the notion of rank is suitably generalized. This generalization is the basic idea of this paper.

Moreover, we investigate the topological structure of $\mathcal{Q}(\mathcal{R})$. It is shown that $\mathcal{Q}(\mathcal{R})$ is a sheaf over the Stone spectrum $\mathcal{Q}(\mathcal{C})$ of the center $\mathcal{C}$ of $\mathcal{R}$. The projection mapping of this sheaf is given by

$$\begin{array}{rccc} \zeta_{\mathcal{C}}: & \mathcal{Q}(\mathcal{R}) & \to & \mathcal{Q}(\mathcal{C}) \\ & \mathfrak{B} & \mapsto & \mathfrak{B} \cap \mathcal{C}. \end{array}$$

This implies that $\mathcal{Q}(\mathcal{R})$ is a locally compact space. The fibres of $\zeta_{\mathcal{C}}$ are discrete and representable as quotients of the unitary group $\mathcal{U}(\mathcal{R})$ of $\mathcal{R}$ modulo a subgroup (depending on the fibre).

In section 5 we solve the “trace-problem” for quasipoints of finite von Neumann algebras $\mathcal{R}$ of type $\mathrm{I_n}$. This problem is the issue whether for a given quasipoint $\mathfrak{B}$ of a von Neumann algebra $\mathcal{R}$ there is a maximal abelian von Neumann subalgebra $\mathcal{M}$ of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M}$ is a quasipoint of $\mathcal{M}$. We show that this is the case for all finite von Neumann algebras of type $\mathrm{I_n}$. Moreover, we show that the property

$$\exists\ \mathfrak{B} \in \mathcal{Q}(\mathcal{R})\ \forall\ \mathcal{A} \in \mathfrak{A}(\mathcal{R}): \ \mathfrak{B} \cap \mathcal{A} \in \mathcal{Q}(\mathcal{A}),$$

where $\mathfrak{A}(\mathcal{R})$ denotes the set of all abelian von Neumann subalgebras of $\mathcal{R}$, is only satisfied in the trivial case that $\mathcal{R}$ is abelian.

2 Local action of partial isometries on Stone spectra

Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$. The following result ([3], [7]) means that the Stone spectrum of the center $\mathcal{C}$ of an arbitrary von Neumann algebra $\mathcal{R}$ is a *quotient* of $\mathcal{Q}(\mathcal{R})$. For the sake of completeness, we present here also the proof given in [3] for the first half of this proposition.

[2] This result was already stated in [7], but the proof contains an error: theorem 33 is not true in general (see remark 3.1)

Proposition 2.1 *Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$ and let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{C}$. Then the mapping*

$$\zeta_{\mathcal{A}} : \mathfrak{B} \mapsto \mathfrak{B} \cap \mathcal{A}$$

is an open continuous, and therefore identifying, mapping from $\mathcal{Q}(\mathcal{R})$ onto $\mathcal{Q}(\mathcal{A})$. Moreover

$$\zeta_{\mathcal{A}}(\mathfrak{B}) = \{s_{\mathcal{A}}(P) \mid P \in \mathfrak{B}\}$$

for all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, where

$$s_{\mathcal{A}}(P) := \bigwedge \{Q \in \mathcal{P}(\mathcal{A}) \mid P \leq Q\}$$

is the $\mathcal{A}$-support of $P \in \mathcal{P}(\mathcal{R})$.
Conversely, if $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$ for all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, then $\mathcal{M}$ is contained in the center of $\mathcal{R}$.

Proof: $\mathfrak{B} \cap \mathcal{A}$ is clearly a dual ideal in $\mathcal{P}(\mathcal{A})$. Let $\beta \in \mathcal{Q}(\mathcal{A})$ be a quasipoint that contains $\mathfrak{B} \cap \mathcal{A}$ and let $C \in \beta$. If $C \notin \mathfrak{B} \cap \mathcal{A}$ then $C \notin \mathfrak{B}$. Hence there is some $P \in \mathfrak{B}$ such that $P \wedge C = 0$. Because C is central this means $PC = 0$. But then $P = PC + P(I - C) = P(I - C)$, i.e. $P \leq I - C$. This implies $I - C \in \mathfrak{B} \cap \mathcal{A} \subseteq \beta$, a contradiction to $C \in \beta$. Hence $\mathfrak{B} \cap \mathcal{A}$ is a quasipoint in $\mathcal{A}$.
It follows immediately from the definition of the $\mathcal{A}$-support that

$$\forall\ P, Q \in \mathcal{P}(\mathcal{R}) :\ P \leq s_{\mathcal{A}}(P) \ \text{ and } \ s_{\mathcal{A}}(P \wedge Q) \leq s_{\mathcal{A}}(P) \wedge s_{\mathcal{A}}(Q)$$

holds. This implies that $\{s_{\mathcal{A}}(P) \mid P \in \mathfrak{B}\}$ is a filter base contained in $\mathfrak{B} \cap \mathcal{A}$. Because of $s_{\mathcal{A}}(P) = P$ for all $P \in \mathcal{P}(\mathcal{A})$, we must have equality.
Now we prove that

(i) $\forall\ P \in \mathcal{P}(\mathcal{R}) :\ \zeta_{\mathcal{A}}(\mathcal{Q}_P(\mathcal{R})) = \mathcal{Q}_{s_{\mathcal{A}}(P)}(\mathcal{A})$ and

(ii) $\forall\ Q \in \mathcal{P}(\mathcal{A}) :\ \overset{-1}{\zeta_{\mathcal{A}}}(\mathcal{Q}_Q(\mathcal{A})) = \mathcal{Q}_Q(\mathcal{R})$

hold: It is obvious that $\zeta_{\mathcal{A}}(\mathcal{Q}_P(\mathcal{R}))$ is contained in $\mathcal{Q}_{s_{\mathcal{A}}(P)}(\mathcal{A})$. Let $\gamma \in \mathcal{Q}_{s_{\mathcal{A}}(P)}(\mathcal{A})$. Then $P \in \overset{-1}{s_{\mathcal{A}}}(\gamma)$, and we shall show that this implies that $\{PQ \mid Q \in \gamma\} \cup \gamma$ is a filter base in $\mathcal{P}(\mathcal{R})$. Since γ consists of central projections, $\{PQ \mid Q \in \gamma\} \cup \gamma$ is a filter base if and only if

$$\forall\ Q \in \gamma :\ PQ \neq 0.$$

Assume that $PQ = 0$ for some $Q \in \gamma$. Then $P \leq I - Q$, hence also $s_{\mathcal{A}}(P) \leq I - Q$, contradicting $s_{\mathcal{A}}(P) \in \gamma$. Let $\mathfrak{B}$ be a quasipoint in $\mathcal{P}(\mathcal{R})$ that contains $\{PQ \mid Q \in \gamma\} \cup \gamma$. Because of $s_{\mathcal{A}}(Q) = Q$ for all $Q \in \gamma$ we obtain

$$\gamma = s_{\mathcal{A}}(\{PQ \mid Q \in \gamma\} \cup \gamma) \subseteq s_{\mathcal{A}}(\mathfrak{B}) = \zeta_{\mathcal{A}}(\mathfrak{B}).$$

Hence $\gamma = \zeta_{\mathcal{A}}(\mathfrak{B})$ since $\zeta_{\mathcal{A}}(\mathfrak{B})$ and γ are quasipoints in $\mathcal{P}(\mathcal{A})$. This proves (i). (ii) follows from the fact that each quasipoint in $\mathcal{P}(\mathcal{A})$ is contained in a quasipoint in $\mathcal{P}(\mathcal{R})$. Properties (i) and (ii) imply that $\zeta_{\mathcal{A}}$ is open, continuous and surjective.
Now let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$ for all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$. Without loss of generality we can assume that $\mathcal{M} = lin_{\mathbb{C}}\{P, I\}$ for some nonzero projection $P \in \mathcal{R}$. Then our condition is equivalent to

$$\forall\ \mathfrak{B} \in \mathcal{Q}(\mathcal{R}) :\ P \in \mathfrak{B}\ \mathit{or}\ I - P \in \mathfrak{B}.$$

Let Q be an arbitrary nonzero projection in $\mathcal{R}$. If $Q \wedge P + Q \wedge (I - P) < Q$, there is a quasipoint $\mathfrak{B}$ of $\mathcal{R}$ that contains $Q - (Q \wedge P + Q \wedge (I - P))$. Then $Q \in \mathfrak{B}$, but $Q \wedge P,\ Q \wedge (I - P) \notin \mathfrak{B}$. According to our condition we have $P \in \mathfrak{B}$ or $I - P \in \mathfrak{B}$, hence $Q \wedge P \in \mathfrak{B}$ or $Q \wedge (I - P) \in \mathfrak{B}$, a contradiction. Thus $Q = Q \wedge P + Q \wedge (I - P)$. Therefore $Q \wedge P = (Q \wedge P)P = QP$, hence $PQ = QP$. Since Q was arbitrary, P is in the center of $\mathcal{R}$. $\square$

If $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ and $F \in \mathfrak{B}$, the set

$$\mathfrak{B}_F := \{P \in \mathfrak{B} \mid P \leq F\}$$

is called the F-socle of $\mathfrak{B}$. It is easy to see that a quasipoint is uniquely determined by any of its socles.

Let $\theta \in \mathcal{R}$ be a partial isometry, i.e. $E := \theta^*\theta$ and $F := \theta\theta^*$ are projections. θ has kernel $E(\mathcal{H})^\perp$ and maps $E(\mathcal{H})$ *isometrically* onto $F(\mathcal{H})$. Now it is easy to see that for any projection $P_U \leq E$ we have

$$\theta P_U \theta^* = P_{\theta(U)}. \tag{1}$$

A consequence of this relation is

$$\forall\ P, Q \leq E :\ \theta(P \wedge Q)\theta^* = (\theta P \theta^*) \wedge (\theta Q \theta^*). \tag{2}$$

If $\mathfrak{B} \in \mathcal{Q}_E(\mathcal{R})$ then

$$\theta_*(\mathfrak{B}_E) := \{\theta P \theta^* \mid P \in \mathfrak{B}_E\} \tag{3}$$

is the F-socle of a (uniquely determined) quasipoint $\theta_*(\mathfrak{B}) \in \mathcal{Q}_F(\mathcal{R})$:
Equation 2 guarantees that $\theta_*(\mathfrak{B}_E)$ is a filter base. Let $\tilde{\mathfrak{B}}$ be a quasipoint that contains $\theta_*(\mathfrak{B}_E)$. Then $\theta_*(\mathfrak{B}_E) \subseteq \tilde{\mathfrak{B}}_F$. Assume that this inclusion is proper. If $Q \in \tilde{\mathfrak{B}}_F \setminus \theta_*(\mathfrak{B}_E)$ then $\theta^* Q\theta \notin \mathfrak{B}_E$ and therefore there is some $P \in \mathfrak{B}_E$ such that $P \wedge \theta^* Q\theta = 0$. But then $\theta P\theta^* \wedge Q = 0$, a contradiction. This shows that we obtain a mapping

$$\begin{array}{rccc} \theta_* : & \mathcal{Q}_{\theta^*\theta}(\mathcal{R}) & \rightarrow & \mathcal{Q}_{\theta\theta^*}(\mathcal{R}) \\ & \mathfrak{B} & \mapsto & \theta_*\mathfrak{B}, \end{array}$$

where $\theta_*\mathfrak{B}$ denotes the quasipoint determined by $\theta_*(\mathfrak{B})$.
It is easy to see that θ_* is a homeomorphism with inverse $(\theta^*)_*$. Note that θ_* is globally defined if θ is given by a unitary operator.

Definition 2.1 *Let $\beta \in \mathcal{Q}(\mathcal{C})$. A quasipoint $\mathfrak{B}$ of $\mathcal{R}$ is called a quasipoint over β if $\mathfrak{B} \cap \mathcal{C} = \beta$ holds. Similarly, $P \in \mathcal{P}(\mathcal{R})$ is called a projection over β if $s_{\mathcal{C}}(P) \in \beta$ holds. We denote by $\mathcal{Q}^\beta(\mathcal{R})$ the set of all quasipoints over β and by $\mathcal{P}^\beta(\mathcal{R})$ the set of all projections over β.*

Lemma 2.1 *Let $\theta \in \mathcal{R}$ be a partial isometry and $\mathfrak{B} \in \mathcal{Q}_{\theta^*\theta}(\mathcal{R})$. Then $\mathfrak{B} \in \mathcal{Q}^\beta(\mathcal{R})$ if and only if $\theta_*\mathfrak{B} \in \mathcal{Q}^\beta(\mathcal{R})$.*

Proof: Let $E := \theta^*\theta \in \mathfrak{B}$ and $F := \theta\theta^*$. Then $F = \theta E\theta^*$, $s_{\mathcal{C}}(E) = s_{\mathcal{C}}(F)$, and the $s_{\mathcal{C}}(E)$-socle $\beta_{s_{\mathcal{C}}(E)} = \{p s_{\mathcal{C}}(E) \mid p \in \beta\}$ is mapped by conjugation with θ onto $\beta_{s_{\mathcal{C}}(E)} = \theta\beta_{s_{\mathcal{C}}(E)}\theta^* \subseteq \gamma_{s_{\mathcal{C}}(E)}$, where $\gamma := \mathcal{C} \cap \theta_*\mathfrak{B}$. If $\beta_{s_{\mathcal{C}}(E)} \neq \gamma_{s_{\mathcal{C}}(E)}$, there is $q \in \gamma$ such that $pq = 0$. But this contradicts the inclusion $\beta_{s_{\mathcal{C}}(E)} \subseteq \gamma_{s_{\mathcal{C}}(E)}$. Hence $\beta_{s_{\mathcal{C}}(E)} = \gamma_{s_{\mathcal{C}}(E)}$, so $\beta = \gamma$. $\square$

The image $\theta_*\mathfrak{B}$ of $\mathfrak{B}$ depends on the partial isometry θ, not only on the projection $\theta^*\theta \in \mathfrak{B}$. If, for example, $\mathcal{R}$ is a factor of type III, every non-zero projection $P \in \mathcal{R}$ is equivalent to I, so $\mathcal{Q}(\mathcal{R}) \cong \mathcal{Q}_P(\mathcal{R})$. The situation is considerably simpler for *abelian* quasipoints.

Definition 2.2 *A quasipoint of $\mathcal{R}$ is called abelian if it contains an abelian projection.*

The term "abelian quasipoint" is motivated by the following fact: If $E \in \mathfrak{B}$ is an abelian projection then the E-socle $\mathfrak{B}_E$, which determines $\mathfrak{B}$ uniquely, consists entirely of abelian projections. Moreover every subprojection of an abelian projection E is of the form CE with a suitable central projection C. Hence

$$\mathfrak{B}_E = \{CE \mid C \in \mathfrak{B} \cap \mathcal{C}\}$$

if E is abelian.

Let $\mathfrak{B}$ be an abelian quasipoint over $\beta \in \mathcal{Q}(\mathcal{C})$ and let $E, F \in \mathfrak{B}$ be abelian projections. Then

$$E \wedge F = pE = qF$$

for suitable $p, q \in \beta$, so $pqE = pqF$. If G is an abelian projection, equivalent to E via the partial isometry θ, then $G \in \mathcal{P}^{\beta}(\mathcal{R})$ and $pqE \sim pqG$ via the partial isometry $pq\theta$. Let $\mathfrak{B}'$ be a quasipoint over β that contains G. If $H \in \mathfrak{B}'$ is any abelian projection, $rG = rH$ for some $r \in \beta$. Therefore, $pqrE \sim pqrH$. Since a quasipoint is determined by any of its socles, it follows that $\mathfrak{B}' = \theta_* \mathfrak{B}$ for every partial isometry $\theta \in \mathcal{R}$ such that $\theta^*\theta \in \mathfrak{B}$, $\theta\theta^* \in \mathfrak{B}'$, and both are abelian. Summing up, we have proved the following result which already appears (with a similar proof) in [7]:

Proposition 2.2 *Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$ and let $\mathfrak{B}, \mathfrak{B}' \in \mathcal{Q}(\mathcal{R})$ be abelian quasipoints. Then $\mathfrak{B}' = \theta_* \mathfrak{B}$ for some partial isometry $\theta \in \mathcal{R}$ if and only if $\mathfrak{B} \cap \mathcal{C} = \mathfrak{B}' \cap \mathcal{C}$.*

3 Rank over a quasipoint of the center

Let $\mathcal{R}$ be a von Neumann algebra of type $\mathrm{I_n}$ ($\mathrm{n} \in \mathbb{N}$) and let $\mathcal{C}$ be the center of $\mathcal{R}$. If $\mathcal{C} = \mathbb{C}$, then $\mathcal{R} \cong \mathcal{L}(\mathbb{C}^n)$ and the rank $rk(P)$ of a projection $P \in \mathcal{R}$ is the maximal number of pairwise orthogonal (equivalent) abelian subprojections of P. In this case, it coincides with the rank of P as a Hilbert space operator. In general, *the rank of $P \in \mathcal{P}(\mathcal{R})$ as a Hilbert space operator is infinite.* If we want to define the rank of P as a natural number, we can do this only *locally* over the quasipoints of $\mathcal{C}$. The definition emerges quite naturally from the following

Theorem 3.1 *([11], Corollary 6.5.5)*
If P and E are projections in a von Neumann algebra $\mathcal{R}$ with center $\mathcal{C}$, $s_{\mathcal{C}}(P) = s_{\mathcal{C}}(E)$, and E is abelian in $\mathcal{R}$, there is a family (p_j) of central projections in $\mathcal{R}$ with sum $s_{\mathcal{C}}(P)$ such that p_jP is the sum of j equivalent abelian projections. If $\mathcal{R}s_{\mathcal{C}}(E)$ is of type $\mathrm{I_n}$, then $1 \leq j \leq n$.

It is useful to throw a short look at the proof of this result to see how the projectionsp_j depend on P.

Let $E \in \mathcal{P}(\mathcal{R})$ be an arbitrary abelian projection with central support I and let $\beta \in \mathcal{Q}(\mathcal{C})$. If $P \in \mathcal{R}$ is a projection over β, the algebra

$\mathcal{R}s_{\mathcal{C}}(P)$ is of type $\mathrm{I_n}$, too, and $s_{\mathcal{C}}(P)E$ is an abelian projection with central support $s_{\mathcal{C}}(P)$. Hence there are mutually orthogonal central projections $p_1, \ldots, p_n$ with sum $s_{\mathcal{C}}(P)$ such that $p_j P$ is the sum of j equivalent abelian projections ($1 \leq j \leq n$). Since $s_{\mathcal{C}}(P) \in \beta$, there is a unique $j_P \in \{1, \ldots, n\}$ such that $p_{j_P} \in \beta$.

Definition 3.1 *Let $P \in \mathcal{R}$ be a projection over $\beta \in \mathcal{Q}(\mathcal{C})$. Then*

$$rk_\beta(P) := j_P$$

is called the rank of P over β.

Note that if $\mathcal{R}$ is a factor, then $\mathcal{Q}(\mathcal{C}) = \{\{I\}\}$ and $rk_{\{I\}}(P) = rk(P)$ for all nonzero $P \in \mathcal{P}(\mathcal{R})$.

Proposition 3.1 *The rank over β has the following properties:*

(i) If $P \in \mathcal{P}^\beta(\mathcal{R})$, then $rk_\beta(pP) = rk_\beta(P)$ for all $p \in \beta$.

(ii) $P \in \mathcal{P}^\beta(\mathcal{R})$ has rank n over β if and only if there is a $p \in \beta$ such that $p \leq P$.

(iii) If $P, Q \in \mathcal{P}^\beta(\mathcal{R})$ and $P \leq Q$, then $rk_\beta(P) \leq rk_\beta(Q)$.

(iv) $P \in \mathcal{P}^\beta(\mathcal{R})$ has rank 1 over β if and only if there is a $p \in \beta$ such that pP is abelian.

(v) If $P \in \mathcal{P}(\mathcal{R}) \setminus \{0\}$, the function $\beta \mapsto rk_\beta(P)$ is defined and locally constant on $\mathcal{Q}_{s_{\mathcal{C}}(P)}(\mathcal{C})$.

Proof: (i) follows directly from elementary results on the equivalence of projections.

(ii) If $rk_\beta(P) = n$, there are mutually orthogonal equivalent abelian projections $E_1, \ldots, E_n$ and $p_n \in \beta$ such that $p_n P = E_1 + \cdots + E_n$. Since $\mathcal{R}$ is of type $\mathrm{I_n}$, there are mutually orthogonal equivalent abelian projections $F_1, \ldots, F_n$ such that $p_n s_{\mathcal{C}}(P) = F_1 + \cdots + F_n$. But $E_1 + \cdots + E_n \sim F_1 + \cdots + F_n$ by [11], proposition 6.2.2, and $p_n P \leq p_n s_{\mathcal{C}}(P) I_n$. Hence $p_n P = p_n s_{\mathcal{C}}(P) = p_n$, because $\mathcal{R}$ is finite and $p_n \leq s_{\mathcal{C}}(P)$.

(iii) Since $P \leq Q$, $s_{\mathcal{C}}(P) \leq s_{\mathcal{C}}(Q)$ and, therefore, $P \leq s_{\mathcal{C}}(P)Q$. Hence we may assume that $s_{\mathcal{C}}(P) = s_{\mathcal{C}}(Q)$. Now, by definition, there

are $p_j, q_k \in \beta$ and orthogonal families $(E_1, \ldots, E_j)$, $(F_1, \ldots, F_k)$ such that $E_1 \sim \cdots \sim E_j$, $F_1 \sim \cdots \sim F_k$ and

$$\begin{aligned} p_j P &= E_1 + \cdots + E_j, \\ q_k Q &= F_1 + \cdots + F_k. \end{aligned}$$

Since $s_{\mathcal{C}}(E_l) = p_j$ and $s_{\mathcal{C}}(F_m) = q_k$ for all $l \leq j,\ m \leq k$, $p_j q_k E_l \sim p_j q_k F_m$ for all $l \leq j,\ m \leq k$. If $k < j$, we deduce from

$$\begin{aligned} p_j q_k P &= p_j q_k E_1 + \cdots + p_j q_k E_j, \\ p_j q_k Q &= p_j q_k F_1 + \cdots + p_j q_k F_k \end{aligned}$$

that $p_j q_k Q$ is equivalent to a proper subprojection of $p_j q_k P$, contradicting $p_j q_k P \leq p_j q_k Q$.

(iv) follows from the very definition of the rank over β.

(v) Clearly, $\beta \mapsto rk_\beta(P)$ is constant on $\mathcal{Q}_{p_j}(\mathcal{C})$. $\square$

Property (i) shows that $rk_\beta(P)$ is merely an invariant of the **set**

$$[P]_\beta := \{pP \mid p \in \beta\}.$$

This leads to the following notion.

Definition 3.2 *Let $\mathcal{R}$ be an arbitrary von Neumann algebra with nontrivial center $\mathcal{C}$, $\beta \in \mathcal{Q}(\mathcal{C})$ and define*

$$\forall\ P, Q \in \mathcal{P}(\mathcal{R}) :\ (P \sim_\beta Q\ :\Longleftrightarrow\ \exists\ p \in \beta :\ pP = pQ).$$

Clearly, since β is a dual ideal, $\sim_\beta$ is an equivalence relation on $\mathcal{P}(\mathcal{R})$. We denote the equivalence class of $P \in \mathcal{P}(\mathcal{R})$ by $[P]_\beta$, or simply by $[P]$, as long as the quasipoint β is fixed.

Remark 3.1 *We can extend $\sim_\beta$ to all of $\mathcal{R}$ by defining*

$$\forall\ A, B \in \mathcal{R} :\ (A \sim_\beta B\ :\Longleftrightarrow\ \exists\ p \in \beta :\ pA = pB).$$

The set $[\mathcal{R}]$ of equivalence classes becomes a $*$- algebra by setting

$$[A]^* := [A^*],\ [A] + [B] := [A + B]\ \ \textit{and}\ \ [A][B] := [AB].$$

For $A \in \mathcal{R}$ let

$$|[A]| := \inf\{|pA| \mid p \in \beta\}.$$

$|[A]|$ is called the β-seminorm of $[A]$. Indeed, the β-seminorm is a submultiplicative seminorm on $[\mathcal{R}]$ that satisfies $|[A]^*[A]| = |[A]|^2$, but, if $\mathcal{C}$ is not trivial, it is not a norm. Moreover, it is easy to prove, using the spectral theorem, that

$$|[a]| = |\tau_\beta(a)|,$$

where τ_β is the character of $\mathcal{C}$ that is induced by β, holds for all $a \in \mathcal{C}$:
It suffices to prove the assertion for $a \geq 0$, since then $|b|_\beta^2 = |b^*b|_\beta = |\tau_\beta(b^*b)| = |\tau_\beta(b)|^2$ for all $b \in \mathcal{C}$.
Let $a \geq 0$. Then for all $p \in \beta$ we have $|\tau_\beta(a)| = |\tau_\beta(pa)| \leq |pa|$, hence

$$|\tau_\beta(a)| \leq |a|_\beta.$$

Let E be the spectral family of a, $\varepsilon > 0$ and $a_\varepsilon := \sum_{k=1}^m \lambda_k(E_{\lambda_k} - E_{\lambda_{k-1}})$ such that $|a - a_\varepsilon| < \varepsilon$, $E_{\lambda_m} = I$, $E_{\lambda_0} = 0$. There is a unique j such that $E_{\lambda_j} - E_{\lambda_{j-1}} \in \beta$. Choose $p \in \beta$ such that $p(E_{\lambda_k} - E_{\lambda_{k-1}}) = 0$ for $k \neq j$. Then

$$\begin{aligned} |pa| &\leq |pa_\varepsilon| + |p(a - a_\varepsilon)| \\ &< |\lambda_j| + \varepsilon \\ &\leq |\tau_\beta(a)| + 2\varepsilon, \end{aligned}$$

so $|a|_\beta \leq |\tau_\beta(a)|$.
Thus, in general, $[\mathcal{C}]$ is an integer domain, but not a field.

We can transfer the partial order of $\mathcal{P}(\mathcal{R})$ to $[\mathcal{P}(\mathcal{R})]$ by

$$\forall\ P, Q \in \mathcal{P}(\mathcal{R}) :\ ([P] \leq [Q]\ :\Longleftrightarrow\ \exists\ p \in \beta :\ pP \leq pQ).$$

This is obviously well defined and satisfies the properties of partial order. Let $P, Q \in \mathcal{P}(\mathcal{R})$ and let

$$[P] \wedge_\beta [Q] := [P \wedge Q].$$

This is well defined, and if $[E] \leq [P]$, $[E] \leq [Q]$, then $[E] \leq [P] \wedge_\beta [Q]$: we have $pE \leq pP$, $pE \leq pQ$ for some $p \in \beta$, hence $pE \leq pP \wedge pQ$, i.e. $[E] \leq [P] \wedge_\beta [Q]$. Thus $[P] \wedge_\beta [Q]$ satisfies the universal property of the minimum.

We return to the discussion of the rank over β. The following result is decisive. It permits to generalize our proof that the quasipoints of a finite *factor* of type I are all atomic, to a proof that the quasipoints of an arbitrary finite von Neumann algebra of type I are all abelian.

Proposition 3.2 *Let $\mathcal{R}$ be a finite von Neumann algebra of type* $\mathrm{I_n}$ *and let $P, Q \in \mathcal{P}(\mathcal{R})$ be projections over $\beta \in \mathcal{Q}(\mathcal{C})$ such that $rk_\beta(P) = rk_\beta(Q)$. Then $[P] \le [Q]$ implies $[P] = [Q]$.*

Proof: Replacing P and Q by $s_{\mathcal{C}}(Q)P$ and $s_{\mathcal{C}}(P)Q$, respectively, we may assume that P and Q have equal central support. Moreover, we can assume that $P \le Q$ holds. Then there are $p \in \beta$ and orthogonal families $(E_1, \dots, E_j)$, $(F_1, \dots, F_j)$ of equivalent abelian projections such that

$$E_1 + \dots + E_j = pP \text{ and } F_1 + \dots + F_j = pQ.$$

Hence $pP = pQ$, since $\mathbb{M}_n(\mathcal{C})$ is finite, and therefore $[P] = [Q]$. $\square$

The rank over a quasipoint of the center shares also another property with the ordinary rank:

Proposition 3.3 *Let $\mathcal{R}$ be a von Neumann algebra of type* $\mathrm{I_n}$ $(\mathrm{n} \in \mathbb{N})$ *and let $P \in \mathcal{P}^\beta(\mathcal{R})$ be a projection with $rk_\beta P < n$. Then $I - P \in \mathcal{P}^\beta(\mathcal{R})$ and $rk_\beta(I - P) = n - rk_\beta P$.*

Proof: For an arbitrary projection $Q \in \mathcal{R}$ let

$$c_{\mathcal{C}}(Q) := \bigvee\{p \in \mathcal{P}(\mathcal{C}) \mid p \le Q\}.$$

Then $c_{\mathcal{C}}(Q) = I - s_{\mathcal{C}}(I - Q)$ and $s_{\mathcal{C}}(I - P) = I - c_{\mathcal{C}}P$ implies that $I - P \in \mathcal{P}^\beta(\mathcal{R})$ if and only if $c_{\mathcal{C}} \notin \beta$. But this is equivalent to $rk_\beta P < n$. Let $k := rk_\beta$ and $m := rk_\beta(I - P)$. Since β is a dual ideal, there is a $p \in \beta$ such that

$$pP = E_1 + \dots + E_k \text{ and } p(I - P) = F_1 + \dots + F_m$$

with abelian projections $E_1, \dots E_k, \dots, F_1, \dots, F_m$. This implies $pI = E_1 + \ldots + E_k + F_1 + \ldots + F_m$, hence $k + m = n$. $\square$

Theorem 3.2 *Let $\mathcal{R}$ be a von Neumann algebra of type* $\mathrm{I_n}$ $(\mathrm{n} \in \mathbb{N})$*. Then all quasipoints of $\mathcal{R}$ are abelian.*

Proof: Let $\mathcal{C}$ be the center of $\mathcal{R}$. Consider a quasipoint $\mathfrak{B}$ of $\mathcal{R}$ and the corresponding quasipoint $\beta := \mathfrak{B} \cap \mathcal{C}$ of $\mathcal{C}$. Let

$$r_0 := \min\{rk_\beta(P) \mid P \in \mathfrak{B}\}$$

and choose $P_0 \in \mathfrak{B}$ with $rk_\beta(P_0) = r_0$. Then we obtain for an arbitrary $P \in \mathfrak{B}$:

$$r_0 \le rk_\beta(P \wedge P_0) \le rk_\beta(P_0) = r_0,$$

i.e.

$$rk_\beta(P \wedge P_0) = rk_\beta(P_0).$$

Hence, by proposition 3.2, $[P \wedge P_0] = [P_0]$. This means

$$\forall\ P \in \mathfrak{B}\ \exists\ p \in \beta:\ pP_0 \leq pP.$$

Since $rk_\beta(P_0) \geq 1$, there is an abelian subprojection E of P_0 with $s_{\mathcal{C}}(E) \in \beta$. If $P \in \mathfrak{B}$, then $pE \leq pP_0 \leq pP$ for a suitable $p \in \beta$. So, in particular, $E \wedge P \neq 0$. Hence, by the maximality of $\mathfrak{B}$, $E \in \mathfrak{B}$. $\square$

Together with the results of section 2, this theorem unveils the structure of the Stone spectrum of a von Neumann algebra of type $\mathrm{I_n}$ ($\mathrm{n} \in \mathbb{N}$).

4 Structure of the Stone spectrum

Lemma 4.1 *If $\mathcal{R}$ is of type* I*, every abelian quasipoint $\mathfrak{B}$ of $\mathcal{R}$ contains an abelian projection with central support I.*

Proof: Let $E \in \mathfrak{B}$ be abelian and let $p := s_{\mathcal{C}}(E)$. Since $\mathcal{R}$ is of type I, there is an abelian $F \in \mathcal{P}(\mathcal{R})$ with central support I. Then

$$G := E + (I - p)F$$

is abelian ([11]), has central support I and, as $E \in \mathfrak{B}$, is contained in $\mathfrak{B}$. $\square$

The proof shows that, for infinite-dimensional $\mathcal{C}$, an abelian quasipoint of a von Neumann algebra of type I contains infinitely many abelian projections with central support I.

We assume from now on that $\mathcal{R}$ is a finite von Neumann algebra of type $\mathrm{I_n}$. According to theorem 3.2, all quasipoints of $\mathcal{R}$ are abelian.

Lemma 4.2 *Let E be an abelian projection over $\beta \in \mathcal{Q}(\mathcal{C})$. Then there is exactly one $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ over β that contains E.*

Proof: Let $\mathfrak{B}, \mathfrak{B}' \in \mathcal{Q}(\mathcal{R})$ be quasipoints over β that contain E. Since $s_{\mathcal{C}}(pE) = ps_{\mathcal{C}}(E)$ for all $p \in \mathcal{P}(\mathcal{C})$, the E-socles of $\mathfrak{B}$ and $\mathfrak{B}'$ are equal to $\{pE \mid p \in \beta\}$. Hence $\mathfrak{B} = \mathfrak{B}'$. $\square$

Corollary 4.1 *Each fibre $\overset{-1}{\zeta_{\mathcal{C}}}(\beta)$ ($\beta \in \mathcal{Q}(\mathcal{C})$) is a discrete subspace of $\mathcal{Q}(\mathcal{C})$ with respect to its relative topology.*

Let E be an abelian projection with central support I. Then E induces a section

$$\sigma_E : \mathcal{Q}(\mathcal{C}) \to \mathcal{Q}(\mathcal{R}).$$

This follows directly from lemma 4.2: $\sigma_E(\beta)$ is defined to be the unique quasipoint over $\beta \in \mathcal{Q}(\mathcal{C})$ that contains E.
More generally, each abelian $E \in \mathcal{P}(\mathcal{R})$ induces, according to lemma 4.2, a section

$$\sigma_E : \mathcal{Q}_{s_{\mathcal{C}}(E)}(\mathcal{C}) \to \mathcal{Q}(\mathcal{R}).$$

It is obvious that $\zeta_{\mathcal{C}} \circ \sigma_E = id_{\mathcal{Q}_{s_{\mathcal{C}}(E)}(\mathcal{C})}$ holds.

Lemma 4.3 *σ_E is continuous.*

Proof: Since $\mathcal{Q}_P(\mathcal{R}) \cap \mathcal{Q}_E(\mathcal{R}) = \mathcal{Q}_{P\wedge E}(\mathcal{R}) = \mathcal{Q}_{pE}(\mathcal{R})$ for some $p \in \mathcal{P}(\mathcal{C})$, we have

$$\overset{-1}{\sigma_E}(\mathcal{Q}_P(\mathcal{R}) \cap \mathcal{Q}_E(\mathcal{R})) = \mathcal{Q}_p(\mathcal{C}) \cap \mathcal{Q}_{s_{\mathcal{C}}(E)}(\mathcal{C}),$$

hence σ_E is continuous. $\square$

The range of σ_E is $\mathcal{Q}_E(\mathcal{R})$. It is the image of the compact set $\mathcal{Q}_{s_{\mathcal{C}}(E)}(\mathcal{C})$ by the continuous mapping σ_E, hence compact, too. This shows that $\mathcal{Q}(\mathcal{R})$ is a *locally compact space*. Moreover, $\zeta_{\mathcal{C}}$, restricted to $\mathcal{Q}_E(\mathcal{R})$, is a homeomorphism onto $\mathcal{Q}_{s_{\mathcal{C}}(E)}(\mathcal{C})$, so $\zeta_{\mathcal{C}} : \mathcal{Q}(\mathcal{R}) \to \mathcal{Q}(\mathcal{C})$ is a local homeomorphism. In other words, $\mathcal{Q}(\mathcal{R})$ is a *sheaf* over $\mathcal{Q}(\mathcal{C})$ with projection mapping $\zeta_{\mathcal{C}}$ ([17], [2]).

In a finite von Neumann algebra $\mathcal{R}$, two projections $P, Q \in \mathcal{R}$ are equivalent if and only if there is a *unitary* $T \in \mathcal{R}$ such that $Q = TPT^*$ ([13]). The unitary group $\mathcal{U}(\mathcal{R})$ operates on $\mathcal{Q}(\mathcal{R})$ by

$$\forall\, T \in \mathcal{U}(\mathcal{R}),\ \mathfrak{B} \in \mathcal{Q}(\mathcal{R}) :\ T.\mathfrak{B} := \{TPT^* \mid P \in \mathfrak{B}\}.$$

Note that this operation is in accordance with the local operation of partial isometries. For, if $F = TET^*$ with $E \in \mathfrak{B}$ and $T \in \mathcal{U}(\mathcal{R})$, $\theta := TE$ is a partial isometry with $\theta^*\theta = ET^*TE = E$, $\theta\theta^* = TET^* = F$ and, if $P \leq E$, $\theta P \theta^* = TEPET^* = TPT^*$. Therefore, $\theta \mathfrak{B}_E \theta^* = (T.\mathfrak{B})_F$, hence $\theta_* \mathfrak{B} = T.\mathfrak{B}$.

It follows from proposition 2.2 and theorem 3.2 that $\mathcal{U}(\mathcal{R})$ operates *transitively* on the fibres of $\zeta_{\mathcal{C}}$. The operation is, in general, not free. The *isotropy group* of $\mathfrak{B} \in \overset{-1}{\zeta_{\mathcal{C}}}(\beta)$,

$$\mathcal{U}(\mathcal{R})_{\mathfrak{B}} := \{T \in \mathcal{U}(\mathcal{R}) \mid T.\mathfrak{B} = \mathfrak{B}\},$$

can easily be determined.

Proposition 4.1 *Let $\mathcal{R}$ be a finite von Neumann algebra of type* $\mathrm{I_n}$, $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ *a quasipoint over* $\beta \in \mathcal{Q}(\mathcal{C})$ *and* $T \in \mathcal{U}(\mathcal{R})$*. Then the following properties of* T *are equivalent:*

(i) $T.\mathfrak{B} = \mathfrak{B}$.

(ii) $TET^* \in \mathfrak{B}$ *for some abelian* $E \in \mathfrak{B}$.

(iii) $TET^* \in \mathfrak{B}$ *for all abelian* $E \in \mathfrak{B}$.

(iv) If $E \in \mathfrak{B}$ *is abelian, there is some* $p \in \beta$ *such that* $pTE = pET$*, i.e.* $[T]_\beta[E]_\beta = [E]_\beta[T]_\beta$ *for all abelian* $E \in \mathfrak{B}$.

Proof: $(iii) \Longrightarrow (ii) \Longrightarrow (i) \Longrightarrow (iii)$ follows immediately from lemma 4.2. If $E \in \mathfrak{B}$ is abelian such that $TET^* \in \mathfrak{B}$, then $[E]_\beta = [TET^*]_\beta$ by proposition 3.2. Thus (ii) implies (iv) and the converse is obvious. $\square$

Because the action of $\mathcal{U}(\mathcal{R})$ is transitive on each fibre of $\zeta_\mathcal{C}$, the fibres are *homogeneous spaces* $\mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{R})_\mathfrak{B}$, where $\mathfrak{B}$ is an arbitrary chosen element of $\overset{-1}{\zeta_\mathcal{C}}(\beta)$. Of course, this representation depends on the choice of $\mathfrak{B}$, but $\mathcal{U}(\mathcal{R})_\mathfrak{B}, \mathcal{U}(\mathcal{R})_{\mathfrak{B}'}$ ($\mathfrak{B}, \mathfrak{B}' \in \overset{-1}{\zeta_\mathcal{C}}(\beta)$) differ only by conjugation with a suitable element of $\mathcal{U}(\mathcal{R})$.

We collect our results in the following

Theorem 4.1 *Let $\mathcal{R}$ be a von Neumann algebra of type* $\mathrm{I_n}$ ($\mathrm{n} \in \mathbb{N}$) *with center $\mathcal{C}$. Then the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of $\mathcal{R}$ is a locally compact space, the projection mapping $\zeta_\mathcal{C} : \mathcal{Q}(\mathcal{R}) \to \mathcal{Q}(\mathcal{C})$ is a local homeomorphism and, therefore, has discrete fibres. The unitary group $\mathcal{U}(\mathcal{R})$ of $\mathcal{R}$ acts transitively on each fibre of $\zeta_\mathcal{C}$. Therefore, each fibre $\overset{-1}{\zeta_\mathcal{C}}(\beta)$ can be represented as a homogeneous space $\mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{R})_\mathfrak{B}$, where the isotropy group $\mathcal{U}(\mathcal{R})_\mathfrak{B}$ of $\mathfrak{B} \in \overset{-1}{\zeta_\mathcal{C}}(\beta)$ is given by*

$$\mathcal{U}(\mathcal{R})_\mathfrak{B} = \{T \in \mathcal{U}(\mathcal{R}) \mid [T]_\beta[E]_\beta = [E]_\beta[T]_\beta \text{ for all abelian } E \in \mathfrak{B}\}.$$

5 Trace of a quasipoint on an abelian von Neumann subalgebra

The foregoing results can be used to answer the following question for the special case of a von Neumann algebra $\mathcal{R}$ of type $\mathrm{I_n}$ ($\mathrm{n} \in \mathbb{N}$).

Problem: Let $\mathfrak{B}$ be a quasipoint of a von Neumann algebra $\mathcal{R}$. Is there a maximal abelian von Neumann subalgebra $\mathcal{M}$ of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M}$ is a quasipoint of $\mathcal{M}$?
$\mathfrak{B} \cap \mathcal{M}$ is called the trace of $\mathfrak{B}$ on $\mathcal{M}$.

This problem is of importance in the presheaf perspective of observables ([6]).

Let $\mathfrak{B}$ be a quasipoint of a von Neumann algebra of type $\mathrm{I_n}$ ($\mathrm{n} \in \mathbb{N}$) and let $\mathcal{C}$ be the center of $\mathcal{R}$. Then $\beta := \mathfrak{B} \cap \mathcal{C}$ is a quasipoint of $\mathcal{C}$. Choose any maximal abelian von Neumann subalgebra $\mathcal{M}'$ of $\mathcal{R}$ and any quasipoint γ' of $\mathcal{M}'$ that contains β. Moreover, let $\mathfrak{B}'$ be a quasipoint of $\mathcal{R}$ that contains γ'. $\mathfrak{B}$ and $\mathfrak{B}'$ are both quasipoints over β, hence there is a $T \in \mathcal{U}(\mathcal{R})$ such that $\mathfrak{B} = T\mathfrak{B}'T^*$. $\mathcal{M} := T\mathcal{M}'T^*$ is then a maximal abelian von Neumann subalgebra of $\mathcal{R}$ and $\gamma := T\gamma'T^*$ is a quasipoint of $\mathcal{M}$ that is contained in $\mathfrak{B}$. We have therefore proved:

Proposition 5.1 *For each quasipoint $\mathfrak{B}$ of a finite von Neumann algebra of type $\mathrm{I_n}$ there is a maximal abelian von Neumann subalgebra $\mathcal{M}$ of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$.*

The next question that appears naturally is, whether $\mathfrak{B} \cap \mathcal{M}$ is a quasipoint of $\mathcal{M}$ for all maximal abelian von Neumann subalgebras of $\mathcal{R}$.
If such a quasipoint would exist, it would induce, according to proposition 2.1, a global section of the spectral presheaf of $\mathcal{R}$, i.e. a family $(\beta_{\mathcal{A}})_{\mathcal{A} \in \mathfrak{A}(\mathcal{R})}$ of quasipoints $\beta_{\mathcal{A}} \in \mathcal{Q}(\mathcal{A})$, where $\mathfrak{A}(\mathcal{R})$ denotes the semilattice of all abelian von Neumann subalgebras of $\mathcal{R}$, with the property that

$$\beta_{\mathcal{A}} = \beta_{\mathcal{B}} \cap \mathcal{A} \textit{ for all } \mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{R}),\ \mathcal{A} \subseteq \mathcal{B}.$$

But, if $\mathcal{R}$ has no direct summand of type I_1 or I_2, an abstract form of the Kochen-Specker theorem ([8], [9]) forbids global sections of the spectral presheaf. We will prove directly that no quasipoint $\mathfrak{B}$ of a finite von Neumann algebra $\mathcal{R}$ of type $\mathrm{I_n}$ ($\mathrm{n} \geq 2$) has the property that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$ for all maximal abelian von Neumann subalgebras of $\mathcal{R}$. In order to show this, it is convenient to represent $\mathcal{R}$ as $\mathbb{M}_n(\mathcal{C})$, the $\mathcal{C}$-algebra of (n,n)-matrices with entries from the center $\mathcal{C}$ of $\mathcal{R}$. $\mathbb{M}_n(\mathcal{C})$ operates on $\mathcal{C}^n$ which we regard as a $\mathcal{C}$-module. (Since $\mathcal{C}$ is abelian, it does not matter whether we regard $\mathcal{C}^n$ as a left or right $\mathcal{C}$-module.) The general structure behind this situation is the theory of C^*-modules ([15]).
If $a = (a_1, \dots, a_n)^t \in \mathcal{C}^n$, the $\mathcal{C}$-linear mapping

$$\begin{array}{rccc} E_a: & \mathcal{C}^n & \to & \mathcal{C}^n \\ & b & \mapsto & (b|a)a, \end{array}$$

where

$$(b|a) := \sum_{k=1}^{n} b_k a_k^*$$

is the $\mathcal{C}$-valued "scalar" product on $\mathcal{C}^n$, is a projection if and only if $(a|a)$ is a projection in $\mathcal{C}$. A short calculation shows that the projection E_a is an abelian projection.

Remark 5.1 *All abelian projections of $\mathbb{M}_n(\mathcal{C})$ are of the form E_a.*

Proof: Choose a "reference-projection", say E_{e_1}, where $e_1 := (1, 0, \dots, 0)^t$. Then E_{e_1} has central support I. If $E \in \mathbb{M}_n(\mathcal{C})$ is an arbitrary abelian projection with central support p, E is equivalent to pE_{e_1}. Then there is a $T \in \mathcal{U}(\mathbb{M}_n(\mathcal{C}))$ such that

$$E = TpE_{e_1}T^* = pE_{Te_1} = E_{pTe_1}. \ \square$$

Now the property that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$ for all maximal abelian von Neumann subalgebras of $\mathcal{R}$, is obviously equivalent to the property

$$\forall\ P \in \mathcal{P}(\mathcal{R}) :\ P \in \mathfrak{B} \text{ or } I - P \in \mathfrak{B}.$$

Lemma 5.1 *The property that $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{T}(M))$ for all maximal abelian von Neumann subalgebras of $\mathcal{R}$, is equivalent to the property*

$$\forall\ P_1, \dots, P_k \in \mathcal{P}(\mathcal{R}) : (P_1 \vee \dots \vee P_k \in \mathfrak{B} \implies \exists\ i \leq k :\ P_i \in \mathfrak{B}).$$

Proof: Let $P_1 \vee \dots \vee P_k \in \mathfrak{B}$, but assume that no P_i belongs to $\mathfrak{B}$. Then, if the first property holds, $I - P_1, \dots, I - P_k \in \mathfrak{B}$ and, therefore,

$$I - P_1 \vee \dots \vee P_k = (I - P_1) \wedge \dots \wedge (I - P_k) \in \mathfrak{B},$$

a contradiction. The converse is obvious. $\square$

Lemma 5.2 *$E_a = E_{a'}$ if and only if there is a unitary $u \in \mathcal{C}$ such that $a' = ua$. u is unique iff $(a|a) = I$.*

Proof: If we consider the elements of $\mathcal{C}$ as continuous functions on the Stone spectrum $\mathcal{Q}(\mathcal{C})$ of $\mathcal{C}$, a unitary is a continuous function $u : \mathcal{Q}(\mathcal{C}) \to S^1$. If $a, a' \in \{b \in \mathcal{C}^n \mid (b|b) \in \mathcal{P}(\mathcal{C})\}$ such that $E_a = E_{a'}$, then $(a|a) = (a'|a')$, since $(a|a)I_n$ is the central support of E_a. Moreover $a' = E_{a'}a' = E_a a' = (a'|a)a$ and, symmetrically, $a = (a|a')a'$. Hence

$$(a|a) = (a'|a') = (a'|a)(a'|a)^*(a|a),$$

which implies $(a'|a)(a'|a)^* = 1$ on the support of $(a|a)$.

$$u(\gamma) := \begin{cases} (a'|a)(\gamma) & \textit{if } \gamma \in S((a|a)) \\ 1 & \textit{otherwise} \end{cases}$$

defines a unitary $u : \mathcal{Q}(\mathcal{C}) \to S^1$ with $a' = ua$. The converse is obvious. $\square$

Let $e := \frac{1}{\sqrt{n}}(1, \ldots, 1)^t$ and $E := E_e$, let further $e_1, \ldots, e_n$ be the "unit vectors" in $\mathcal{C}^n$ and let $E_k := E_{e_k}$ $(k = 1, \ldots, n)$. The projections $E, E_1, \ldots, E_n$ are abelian and have central support I. Since the quasipoints $T\mathfrak{B}T^*$ $(T \in \mathcal{U}(\mathcal{R}))$ have the property that $T\mathfrak{B}T^* \cap \mathcal{M} \in \mathcal{Q}(\mathcal{T}(M))$ for all maximal abelian von Neumann subalgebra $\mathcal{M}$ if and only if $\mathfrak{B}$ has this property, we can assume that $E \in \mathfrak{B}$. Because of $E_1 + \cdots + E_n = I \in \mathfrak{B}$, we conclude that $E_k \in \mathfrak{B}$ for some $k \leq n$. Since E and E_k are abelian and have central support I, there is a $p \in \mathfrak{B} \cap \mathcal{C}$ such that $pE_k = pE$. But, applying the foregoing lemma, this leads to a contradiction, since the components e_{kj} of e_k are zero for $j \neq k$, while the corresponding components of e are equal to $\frac{1}{\sqrt{n}}$. Hence we have proved

Proposition 5.2 *Let $\mathcal{R}$ be a finite von Neumann algebra of type* $\mathrm{I_n}$*. The following properties are equivalent:*

(i) $n = 1$*, i.e.* $\mathcal{R}$ *is abelian,*

(ii) $\exists\, \mathfrak{B} \in \mathcal{Q}(\mathcal{R})\ \forall\, P \in \mathcal{P}(\mathcal{R}) :\ P \in \mathfrak{B}$ *or* $I - P \in \mathfrak{B}$*,*

(iii) $\forall\, \mathfrak{B} \in \mathcal{Q}(\mathcal{R})\ \forall\, P \in \mathcal{P}(\mathcal{R}) :\ P \in \mathfrak{B}$ *or* $I - P \in \mathfrak{B}$*.*

Together with the abstract Kochen-Specker theorem we get

Corollary 5.1 *If $\mathcal{M}$ is a maximal abelian von Neumann subalgebra of a non-abelian von Neumann algebra $\mathcal{R}$, there is a quasipoint $\mathfrak{B}$ of $\mathcal{R}$ such that $\mathfrak{B} \cap \mathcal{M} \notin \mathcal{Q}(\mathcal{M})$.*

If $\mathcal{M}$ is a maximal abelian von Neumann subalgebra of a von Neumann algebra $\mathcal{R}$, a quasipoint $\mathfrak{B}$ of $\mathcal{R}$ is called *admissible* for $\mathcal{M}$, if $\mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})$. The foregoing results promise that the study of the sets

$$\mathcal{Q}(\mathcal{R})_{\mathcal{M}} := \{\mathfrak{B} \in \mathcal{Q}(\mathcal{R}) \mid \mathfrak{B} \cap \mathcal{M} \in \mathcal{Q}(\mathcal{M})\}$$

and

$$\mathfrak{B}_{\mathfrak{A}(\mathcal{R})} := \{\mathcal{A} \in \mathfrak{A}(\mathcal{R}) \mid \mathfrak{B} \cap \mathcal{M} \notin \mathcal{Q}(\mathcal{M})\}$$

will be an interesting task.

References

[1] G. Birkhoff: Lattice theory,
AMS Coll. Publ., 1974

[2] F. Constantinescu, H.F. de Groote: Geometrische und algebraische Methoden der Physik - Supermannigfaltigkeiten und Virasoro-Algebren, Teubner Verlag, 1994

[3] H.F. de Groote: Observables I : Stone Spectra,
arXiv: math-ph/05 09 020

[4] H.F. de Groote: Observables II : Quantum Observables,
arXiv: math-ph/05 09 075

[5] H.F. de Groote: Observables III : Classical Observables,
arXiv: math-ph/06 01 011

[6] H.F. de Groote: Observables IV : The Presheaf Perspective,
in preparation (2006)

[7] A. Döring: Stone spectra of von Neumann algebras of type I_n,
arXiv: math.OA/05 01 278

[8] A. Döring: Kochen-Specker Theorem for von Neumann Algebras,
arXiv: quant-ph/04 08 106 and International Journal of Theoretical Physics Vol. 44, No. 2 (2005), 139-160.

[9] J. Hamilton, C.J. Isham, J. Butterfield: A topos perspective on the Kochen-Specker theorem: III. Von Neumann algebras as the base category,
Int. J. Theor. Phys. **39**(2000), 1413-1436

[10] R.V. Kadison and J.R. Ringrose: Fundamentals of the Theory of Operator Algebras, Vol. I (Elementary Theory)
Am.Math.Soc. 1997

[11] R.V. Kadison and J.R. Ringrose: Fundamentals of the Theory of Operator Algebras, Vol. II (Advanced Theory)
Am.Math.Soc. 1997

[12] R.V. Kadison and J.R. Ringrose: Fundamentals of the Theory of Operator Algebras, Vol. III Special Topics (Elementary Theory)
Birkhäuser Boston 1991

[13] R.V. Kadison and J.R. Ringrose: Fundamentals of the Theory of Operator Algebras, Vol. IV Special Topics (Advanced Theory)
Birkhäuser Boston 1992

[14] G. Kalmbach: Orthomodular Lattices
L.M.S. Monographs, Academic Press 1983

[15] E.C. Lance: Hilbert C^*-modules - A toolkit for operator algebraists, LMS LN 210, Cambridge University Press 1995

[16] M.H. Stone: The theory of representations for Boolean algebras, Trans. AMS **40** (1936), 37-111

[17] F.W. Warner: Foundations of differentiable manfolds and Lie groups, GTM 94, Springer Verlag 1983

Hans F. de Groote
degroote@math.uni-frankfurt.de
Institut für Analysis und Mathematische Physik
FB Informatik und Mathematik
J.W. Goethe Universität
Frankfurt a.M.
Germany